Mathematics
Higher level
Paper 3 – discrete mathematics

Thursday 21 May 2015 (afternoon)

1 hour

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- Answer all the questions.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A graphic display calculator is required for this paper.
- A clean copy of the **mathematics HL and further mathematics HL formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[60 marks]**.

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Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 14]

(a) The weights of the edges of a graph H are given in the following table.

	A	B	C	D	E	F	G
A	–	5	4	–	–	–	–
B	5	–	–	–	5	–	–
C	4	–	–	5	2	–	–
D	–	–	5	–	3	–	6
E	–	5	2	3	–	5	4
F	–	–	–	–	5	–	1
G	–	–	–	6	4	1	–

(i) Draw the weighted graph H.

(ii) Use Kruskal's algorithm to find the minimum spanning tree of H. Your solution should indicate the order in which the edges are added.

(iii) State the weight of the minimum spanning tree. [8]

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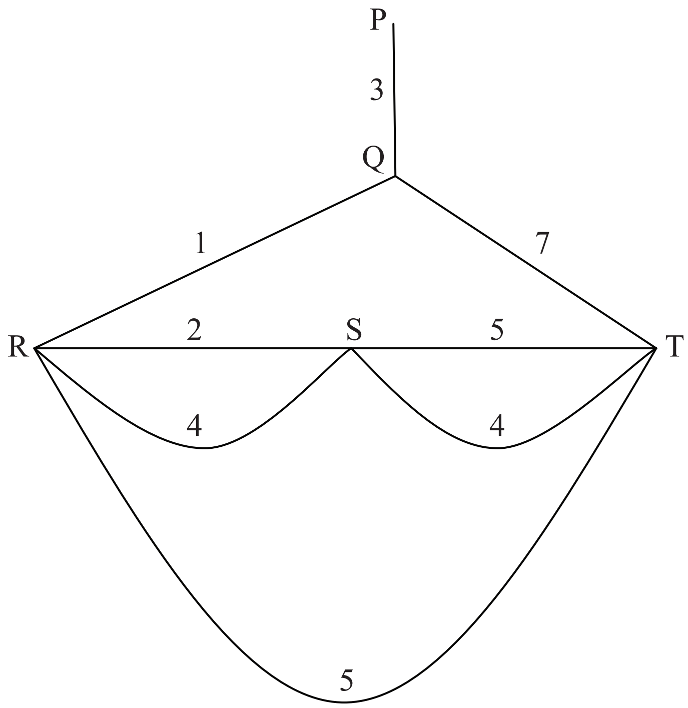
(Question 1 continued)

(b) Consider the following weighted graph.



(i) Write down a solution to the Chinese postman problem for this graph.

(ii) Calculate the total weight of the solution. [3]

(c) (i) State the travelling salesman problem.

(ii) Explain why there is no solution to the travelling salesman problem for this graph. [3]

2. [Maximum mark: 7]

The graph $K_{2,2}$ is the complete bipartite graph whose vertex set is the disjoint union of two subsets each of order two.

(a) Draw $K_{2,2}$ as a planar graph. [2]

(b) Draw a spanning tree for $K_{2,2}$. [1]

(c) Draw the graph of the complement of $K_{2,2}$. [1]

(d) Show that the complement of any complete bipartite graph does not possess a spanning tree. [3]

3. [Maximum mark: 16]

The sequence $\{u_n\}$, $n \in \mathbb{N}$, satisfies the recurrence relation $u_{n+1} = 7u_n - 6$.

(a) Given that $u_0 = 5$, find an expression for u_n in terms of n. [5]

The sequence $\{v_n\}$, $n \in \mathbb{N}$, satisfies the recurrence relation $v_{n+2} = 10v_{n+1} + 11v_n$.

(b) Given that $v_0 = 4$ and $v_1 = 44$, find an expression for v_n in terms of n. [7]

(c) Show that $v_n - u_n \equiv 15 \pmod{16}$, $n \in \mathbb{N}$. [4]

4. [Maximum mark: 12]

A simple connected planar graph, has e edges, v vertices and f faces.

(a) (i) Show that $2e \geq 3f$ if $v > 2$.

(ii) Hence show that K_5, the complete graph on five vertices, is not planar. [6]

(b) (i) State the handshaking lemma.

(ii) Determine the value of f, if each vertex has degree 2. [4]

(c) Draw an example of a simple connected planar graph on 6 vertices each of degree 3. [2]

5. [Maximum mark: 11]

(a) State the Fundamental theorem of arithmetic for positive whole numbers greater than 1. [2]

(b) Use the Fundamental theorem of arithmetic, applied to 5577 and $99\,099$, to calculate $\gcd(5577, 99\,099)$ and $\text{lcm}(5577, 99\,099)$, expressing each of your answers as a product of prime numbers. [3]

(c) Prove that $\gcd(n, m) \times \text{lcm}(n, m) = n \times m$ for all $n, m \in \mathbb{Z}^+$. [6]